RESTRICTIONS ON POSSIBLE FORMS OF CLASSICAL MATTER FIELDS CARRYING NO ENERGY

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It is postulated in general relativity that the matter energy-momentum tensor vanishes if and only if all the matter fields vanish. In classical Lagrangian field theory the energy and momentum density are described by the variational (symmetric) energy-momentum tensor (named the stress tensor) and *a priori* it might occur that for some systems the tensor is identically to zero for all field configurations whereas evolution of the system is subject to deterministic Lagrange equations of motion. Such a system would not generate its own gravitational field. To check if these systems can exist in the framework of classical field theory we find a relationship between the stress tensor and the Euler operator (*i.e.* the Lagrange field equations). We prove that if a system of interacting scalar fields (the number of fields cannot exceed the spacetime dimension d) or a single vector field (in spacetimes with d even) has the stress tensor such that its divergence is identically zero (*i.e.* "on and off shell"), then the Lagrange equations of motion hold identically too. These systems have then no propagation equations at all and should be regarded as unphysical. Thus nontrivial field equations require the stress tensor be nontrivial too. This relationship between vanishing (of divergence) of the stress tensor and of the Euler operator breaks down if the number of fields is greater than d. We show on concrete examples that a system of $n > d$ interacting scalars or two interacting vector fields can have the stress tensor equal identically to zero while their propagation equations are nontrivial. This means that non-self-gravitating (and yet detectable) field systems are in principle admissible. Their equations of motion are, however, in some sense degenerate. We also show, that for a system of arbitrary number of interacting scalar fields or for a single vector field (in some specific spacetimes in the latter case), if the stress tensor is not identically zero, then it cannot vanish for all solutions. There do exist solutions with nonzero energy density and the system back-reacts on the spacetime.

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1. Introduction

It is commonly accepted in general relativity that the gravitational field is generated by all forms of matter (*i.e.* all species of elementary particles and fields) and that all features of a given form of matter that are relevant for determining its gravitational field are encoded in its energy-momentum tensor (*cf. e.g.* [1, 2]). In other terms energy and linear momentum are the source of the gravitational field. This postulate does not tell one how to construct the energy-momentum tensor for a given kind of matter and whether it is unique. In principle this tensor should be determined in a special-relativistic theory describing the material system and then minimally coupled to gravity. However, a reliable expression for this tensor has been found only in few cases (classical electrodynamics, relativistic hydrodynamics *etc.*) and it has turned out that the canonical energy-momentum tensor, though being conceptually important due to the first Noether theorem, does not provide in most cases the correct value of energy density and therefore is unphysical. A unique universal definition of energy and momentum density arises if the equations of motion for the matter under consideration can be derived from a Lagrangian. This is the Hilbert variational (with respect to the spacetime metric) energy-momentum tensor (or Belinfante tensor), hereafter denoted as the stress tensor. It is worth emphasizing that the physical energy-momentum tensor cannot be determined by merely manipulating with the matter equations of motion and the use of the stress tensor is indispensable [3]. Now it is commonly accepted that it is this tensor that correctly describes energy and momentum density and their flows for any Lagrangian matter both in curved and flat spacetimes [4].

It is postulated that the Lagrangian for any matter is such that the resulting stress tensor $T_{\mu\nu}$ vanishes on an open domain in the spacetime if and only if the matter fields vanish on the domain [1]. This condition expresses the principle that any matter carries energy. The energy conditions ([1], Chap. 4) actually imposed on a matter Lagrangian exclude negative energies and ensure the "only if" condition.

In this paper we investigate the problem whether there is some classical matter having no energy at all, *i.e.* whether there exists a matter Lagrangian giving rise to the stress tensor vanishing identically, $T_{\mu\nu} \equiv 0$, for all field configurations independently of the field equations ("on and off shell"). Such a matter should be subject to deterministic equations of motion (the Lagrange ones) and according to Einstein field equations would propagate as a test one in a fixed spacetime. This matter would be non-self-gravitating and its interactions with other fields would be severely restricted by the constraint that they should exclude any energy transfer. At first sight one might conclude that the lack of any energy exchange with ordinary matter systems implies that this matter is nondetectable and as such it may be merely ignored. This is the case in quantum theory. However, here we are dealing with classical fields and in classical physics any system may be treated as open, *i.e.* as being in contact with some surrounding. The contact may be arbitrarily weak, *i.e.* involve negligibly small energy transfer (or no transfer at all), nevertheless by observing the surrounding (which acts as a "marker") one can make measurements on the system. In Example 1 we give a hint of how a system of scalar fields without energy might be detected by their coupling to the electromagnetic field. We do not pursue the problem further, it is sufficient to say here that the lack of energy does not imply "physical nonexistence" in the sense of nondetectability.

This is why we investigate in this work to what extent Lagrangian field theory admits such bizarre systems and whether some of them can be excluded on theoretical grounds. To this end we show for some classical fields that there is a close connection between energy (the stress tensor) and deterministic equations of motion. For these fields we prove that the identically vanishing stress tensor implies that the Lagrange equations also identically vanish for all field configurations, *i.e.* the fields are subject to no propagation equations at all. On very generic physical grounds one can then reject these fields as unphysical.

We first formulate the problem in full generality. Consider a system of classical matter tensor fields ψ_A with a collective index A (there is no need to deal with spinor fields since they can be expressed as tensor ones [1]). An example of a vanishing stress tensor is provided by a Lagrangian for ψ_A of the form

$$L(\psi_A, \psi_{A;\mu}) = \frac{1}{\sqrt{-g}} L_0(\psi_A, \psi_{A,\mu}) \,, \tag{1}$$

where L_0 is a scalar density of the weight $+1$ (so that L is a genuine scalar) and L_0 is independent of the spacetime metric and its first derivatives,

$$\frac{\partial L_0}{\partial g_{\mu\nu}} = \frac{\partial L_0}{\partial g_{\mu\nu,\alpha}} = 0 \,. \tag{2}$$

(Here $\psi_{A;\mu} \equiv \nabla_\mu \psi_A$ is the covariant derivative w.r.t. $g_{\mu\nu}$.) Then the stress tensor (the signature is $-+++$)

$$T_{\mu\nu}(\psi_A) \equiv \frac{-2}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} (L\sqrt{-g}) \tag{3}$$

vanishes identically and the spacetime evolves, in absence of other forms of matter, as an empty one, $G_{\mu\nu} = 0$. We shall consider in the next section a few specific forms of L_0 for scalar and vector fields and find in which cases the corresponding Lagrange equations are trivial, making the fields unphysical. However it is clear that the conditions (1) and (2) are merely a (restrictive) sufficient condition for having $T_{\mu\nu} \equiv 0$ and not a necessary one. In fact, there are altogether 50 conditions (2) whereas there are only 10 identities $T_{\mu\nu}(\psi_A) \equiv 0$ actually imposed on a possible Lagrangian. In what follows, we shall assume in Propositions 1 to 4 that the field Lagrangian $L(\psi_A, \psi_{A;\mu}, g_{\mu\nu})$ may depend on the metric both explicitly and implicitly (via the covariant derivatives) and only in Examples 1 to 5 we shall consider various Lagrangians satisfying Eqs. (1) and (2).

To avoid any confusion we emphasize that we work in the framework of classical field theory and thus we do not take into account the classical topological field theories [5] such as BF theory [6]. They are metric independent and in this sense they might seem relevant to the present work, but their actions are typically given by surface integrals and they describe global observables related to the topological invariants of the manifolds on which they are defined. These theories have no local degrees of freedom, so there are no propagating field excitations (particles).

One may expect that if the system under consideration consists of a large number of fields, then vanishing of the full stress tensor will turn out to be a condition too weak to make trivial the large number of (coupled) Lagrange equations for the fields. We shall show on concrete examples that this is the case. Thus, we should theoretically allow for some specific systems of fields which carry no energy nor momentum, which nonetheless obey deterministic equations of motion (causal or not). The purpose of the present work is to show how many fields and of what type are necessary to this aim and how peculiar their Lagrangians must be.

The main thrust of the paper are Propositions 1 and 2 in Sec. 2 and Propositions 3 and 4 of Sec. 3. Propositions 1 and 2 state that if a classical system consists either of a number of scalar fields or a single vector field and the stress tensor for the system is zero on and off shell, then Lagrange equations hold identically. This system is thus regarded as unphysical. However, if the number of scalar fields exceeds the dimensionality of the spacetime or there are two (or more) vector fields, the theorems break down. It is also relevant whether the dimensionality is even or odd. We show on specific examples that for sufficiently large number of fields (or their components) there exist Lagrangians satisfying conditions (1) and (2) and possessing the Lagrange equations of motion. Their dynamics is however quite bizarre: for systems of interacting fields there are no free-field solutions, for some cases the systems of propagation equations are either degenerate (of first order) or indeterministic (less equations than degrees of freedom). One concludes that classical fields without energy, though not excluded by principles of Lagrange field theory, require very peculiar and easily recognizable Lagrangians and these are unlikely to appear in modelling the physical reality.

Propositions 3 and 4 of Sec. 3 solve, for interacting scalar fields or a single vector field, the problem of whether the stress tensor may vanish for all solutions while it is nonzero for some field configurations off shell. It turns out that there are always some solutions for which the stress tensor cannot vanish. In this sense nontrivial equations of motion imply in most cases a back reaction of the system on the spacetime. Unfortunately, for a vector field the proof works only in a small neighbourhood of flat spacetime and can be generalized solely to spacetimes admitting a covariantly constant Killing vector.

In Section 4 we make some critical comments on the canonical energy-momentum tensor. Proofs of Propositions 1 and 2 employ the second Noether theorem and Proposition 4 requires the Belinfante–Rosenfeld identity. A detailed derivation of both the identities is provided in the Appendix.

2. Scalar and vector fields having no energy

According to Introduction one should not expect to eliminate as unphysical the systems carrying no energy if they involve many interacting scalar and vector fields or fields with high spins. Instead one should separately investigate systems involving rather a small number of scalar fields or a single vector field. These fields are defined on a curved spacetime and are viewed either as test fields or as a part of a larger matter source of gravity. No specific gravitational field equations are assumed. The two systems are dealt with in the following two propositions. Let $d \geq 3$ be the dimensionality of the spacetime.

Proposition 1. Let a material system consist of $n \leq d$ interacting scalar fields and let the divergence of the stress tensor vanish identically for all values of the fields, $T^{\mu\nu}{}_{;\nu} \equiv 0$. Then their Lagrange equations of motion also hold identically for all values of the fields.

Proposition 2. Let the dimension d be even and let a material system consist of a single vector field A_μ with a Lagrangian which can be expanded in a Taylor series in $A_{[\mu;\nu]}$ in the function space of all antisymmetric tensor fields $F_{\mu\nu} = -2A_{[\mu;\nu]}$ defined on an open domain in the spacetime. (The series is centered at $F_{\mu\nu} = 0$ and thus the Lagrangian and all its derivatives are regular at this point.) If the divergence of the stress tensor vanishes identically for all values of the vector field, $T^{\mu\nu}{}_{;\nu} \equiv 0$, then the equations of motion hold identically. Moreover, in $d = 4$ the stress tensor itself is zero, $T^{\mu\nu}(A) \equiv 0$.

Idea of the proof

The proof of both Propositions is based on the Noether identity, valid for any matter field, which arises from the coordinate invariance of the matter action integral. The identity is derived in the Appendix. It involves the divergence $T^{\mu\nu}{}_{;\nu}$ rather than the stress tensor itself and this is why the assumption of Propositions is apparently weaker than $T_{\mu\nu} \equiv 0$. At least in the case of a vector field and $d = 4$ the two assumptions are equivalent.

I. Proof of Proposition 1. Consider a system of n interacting scalar fields ϕ_a, $a = 1, \ldots, n$. For scalars the coefficients $Z_A{}^\beta{}_\alpha$ introduced in Appendix, Eq. (A.15), are zero and the Noether identity (A.17) reduces to

$$E^a \phi_{a;\mu} = T_{\mu\nu}{}^{;\nu}, \tag{4}$$

where $T_{\mu\nu}$ depends on all the scalars and their first and second (in the case of a nonminimal coupling) order derivatives and the same holds for the n scalar quantities E^a. (If $T_{\mu\nu}$ does involve $\phi_{a;\mu\nu}$, then the terms containing third order derivatives, arising on the r.h.s. of (4), do cancel each other.) For $n \leq d$ and arbitrary fields ϕ_a, the vectors $\phi_{a;\mu}$ are linearly independent. On the other hand the condition $T_{\mu\nu}{}^{;\nu} \equiv 0$ gives $E^a \phi_{a;\mu} \equiv 0$ suggesting that the gradients are actually linearly dependent. The consistency is restored only if $E^a \equiv 0$, *i.e.* the field equations hold trivially.

The theorem breaks down for $n > d$ as is seen from the following example[1]. (Our experience with some readers shows that it should be explicitly stated that the determinant of any mixed tensor $Y^\mu{}_\nu$ is an absolute scalar, hence $|\det(Y_{\mu\nu})|^{1/2}$ is a scalar density, *i.e.* transforms as $\sqrt{-g}$. Accordingly, the Lagrangians in the Examples 1, 4 and 5 are absolute scalars. On the other hand the Lagrangians in the Examples 2 and 3 are chosen as pseudoscalars (*i.e.* they transform as scalars multiplied by $J/|J|$, where J is the Jacobian of a coordinate transformation) merely for computational simplicity and can be made scalars by taking the absolute value; this change will not affect the conclusions which follow from them.)

Example 1

For a system of n scalar fields one defines $P_{\mu\nu} \equiv \sum_{a=1}^{n} \phi_{a,\mu}\phi_{a,\nu}$. For arbitrary scalars and $n \geq d$ its determinant $\det(P_{\mu\nu}) \neq 0$ and may be used to make up a Lagrangian, specifically,

$$L_\phi = \frac{1}{\sqrt{-g}} \left|\det(P_{\mu\nu})\right|^{1/2}. \tag{5}$$

Then the conditions (1)–(2) hold and $T_{\mu\nu} \equiv 0$. Let $d = 4$ for simplicity.

(i) For $n = 4$ the Lagrangian takes on a simpler form,

$$L_\phi = \frac{1}{\sqrt{-g}} \left|\det(\phi_{a,\mu})\right|$$

[1] Examples 1 and 4 were suggested to the author by Andrzej Staruszkiewicz.

and furthermore it is a full divergence,

$$L_\phi = \left|\nabla_\alpha(\varepsilon^{\alpha\beta\mu\nu}\phi_1\phi_{2,\beta}\phi_{3,\mu}\phi_{4,\nu})\right|, \tag{6}$$

where $\varepsilon^{\alpha\beta\mu\nu}$ is the antisymmetric Levi–Civita pseudotensor (*i.e.* it transforms as a tensor times $J/|J|$) with $\varepsilon^{0123} = 1/\sqrt{-g}$. Clearly then $E^a(\phi_b) \equiv 0$. *(ii)* Yet in the case of $n = 5$ scalars the Lagrangian (5) cannot be simplified to an analogous form and furthermore it is not a divergence. The five Euler operators do not vanish,

$$\begin{aligned} E^a(\phi_b) &\equiv \frac{\partial L_\phi}{\partial \phi_a} - \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\frac{\partial L_\phi}{\partial \phi_{a,\mu}}\right) \\ &= L_\phi Q^{\alpha\beta}\left(Q^{\mu\nu}\phi_{a,\nu}\sum_{b=1}^{5}\phi_{b,\mu}\phi_{b,\alpha\beta} - \phi_{a,\alpha\beta}\right), \end{aligned} \tag{7}$$

where $Q^{\alpha\beta}$ is the symmetric inverse of $P_{\mu\nu}$, $Q^{\mu\alpha}P_{\alpha\nu} \equiv \delta^\mu_\nu$. To show that E^a are not identically zero one finds counterexamples. In Minkowski spacetime one puts four of ϕ_a equal to the Cartesian coordinates and the fifth scalar equal to some nonlinear functions of time, *e.g.* t^2 or e^t. Then two of the operators are different from zero. One concludes that the system of 5 scalar fields with the Lagrangian (5) is a form of non-self-gravitating and carrying no energy matter, subject to nonlinear propagation equations. The relationship $E^a\phi_{a,\mu} \equiv 0$ shows that there are at most four independent equations for the five scalars, thus the equations of motion and the initial data do not uniquely determine the evolution of the system.

The system might be detected, at least in principle, by its influence on the electromagnetic field A_μ. Let $j^\mu \equiv q\sum_{a=1}^{5} Q^{\mu\nu}\phi_{a,\nu}$, where q is a coupling constant, be a current associated with the scalars. The current is coupled to the electromagnetic potential via the standard interaction term, $L_{\mathrm{int}} = j^\mu A_\mu$, and the stress tensor corresponding to the Lagrangian $L_\phi + L_{\mathrm{int}}$ remains identically zero. Yet the electromagnetic field is affected since the current enters the Maxwell equations.

II. Proof of Proposition 2. Here the main idea of the proof consists in expanding the Noether identity for the vector field, in the case where $T^{\mu\nu}{}_{;\nu} \equiv 0$, in a series of identities (not equations) ultimately resulting in the Euler operator which vanishes identically.

For a single vector field the Noether identity (A.17) takes the form (A.18),

$$E^\mu F_{\alpha\mu} \equiv T_{\alpha\nu}{}^{;\nu} + A_\alpha E^\mu{}_{;\mu} \tag{8}$$

with $F_{\alpha\beta} \equiv A_{\beta;\alpha} - A_{\alpha;\beta}$ being a "field strength" and the Euler operator

$$E^\mu[L(A)] \equiv \frac{\partial L}{\partial A_\mu} - \nabla_\alpha\left(\frac{\partial L}{\partial A_{\mu;\alpha}}\right). \tag{9}$$

We set $T_{\alpha\nu}{}^{;\nu} \equiv 0$ and decompose the identity (8) into a sum of terms containing covariant derivatives of A_μ of definite order. To this aim we first introduce tensors

$$B^{\mu\gamma\alpha\beta} \equiv \frac{\partial^2 L}{\partial A_{\mu;\gamma}\partial A_{\alpha;\beta}} = B^{\alpha\beta\mu\gamma}\,, \tag{10}$$

$$d^{\alpha\beta} \equiv \frac{\partial L}{\partial A_{\alpha;\beta}} \qquad \text{and} \qquad l^\mu \equiv \frac{\partial L}{\partial A_\mu}, \tag{11}$$

these are functions of A_μ and $A_{\mu;\nu}$ (and the metic). Then one finds

$$E^\mu = -B^{\mu\gamma\alpha\beta}A_{\alpha;\beta\gamma} - \frac{\partial d^{\mu\alpha}}{\partial A_\nu}A_{\nu;\alpha} + l^\mu. \tag{12}$$

We insert this expression for E^μ into (8) and decrease the order of derivatives of A_μ with the aid of Ricci identity. After some manipulations the identity $E^\mu F_{\alpha\mu} - A_\alpha E^\mu{}_{;\mu} \equiv 0$ takes on the following involved form,

$$\begin{aligned} 0 \equiv\; & A_\alpha B^{\mu\nu\beta\gamma}A_{\beta;\gamma(\mu\nu)} + A_\alpha\frac{\partial B^{\mu\gamma\nu\beta}}{\partial A_{\lambda;\sigma}}A_{\lambda;(\mu\sigma)}A_{\nu;(\beta\gamma)} \\ & + A_{\lambda;(\mu\sigma)}\left\{-F_{\alpha\nu}B^{\nu\sigma\lambda\mu} + A_\alpha\left[A^\tau R_{\tau\nu\beta\gamma}\frac{\partial B^{\lambda\sigma(\mu\gamma)}}{\partial A_{\nu;\beta}} + 2\frac{\partial B^{(\mu\nu)\lambda\sigma}}{\partial A_\beta}\right.\right. \\ & \left.\left. + \frac{\partial d^{(\mu\sigma)}}{\partial A_\lambda} - \frac{\partial d^{\lambda\sigma}}{\partial A_\mu}\right]\right\} + a_\alpha(A_\mu, A_{\mu;\nu}, R_{\mu\nu\lambda\sigma})\,, \end{aligned} \tag{13}$$

where a_α is a complicated term containing no higher derivatives than of first order. Being an identity for arbitrary A_μ, the terms with derivatives of different order cannot cancel each other, instead they should vanish separately. This implies that the identity splits into a cascade of four sets of independent identities. We first consider the term with third derivatives. It vanishes if their coefficients are zero. Since $A_\alpha \neq 0$ one gets $B^{(\mu\nu)\beta\gamma} \equiv 0$, then the definition (10) yields

$$B^{\alpha\beta\mu\nu} = B^{\mu\nu\alpha\beta} = -B^{\beta\alpha\mu\nu} = -B^{\alpha\beta\nu\mu}. \tag{14}$$

These in turn imply a specific relationship between $B^{\alpha\beta\mu\nu}$ and $d^{\alpha\beta}$. It turns out useful to decompose $A_{\alpha;\beta}$ and $d^{\alpha\beta}$ into symmetric and antisymmetric parts, $A_{\alpha;\beta} = A_{(\alpha;\beta)} + \frac{1}{2}F_{\beta\alpha}$ and

$$d^{(\alpha\beta)} = \frac{\partial L}{\partial A_{(\alpha;\beta)}} \equiv S^{\alpha\beta}, \qquad d^{[\alpha\beta]} = 2\frac{\partial L}{\partial F_{\beta\alpha}} \equiv N^{\alpha\beta}. \tag{15}$$

Since

$$B^{\mu\nu\alpha\beta} = \frac{\partial d^{\alpha\beta}}{\partial A_{\mu;\nu}} = \frac{\partial d^{\mu\nu}}{\partial A_{\alpha;\beta}}$$

one finds

$$B^{\alpha\beta\mu\nu} = \frac{\partial S^{\alpha\beta}}{\partial A_{(\mu;\nu)}} + \frac{\partial N^{\alpha\beta}}{\partial A_{(\mu;\nu)}} - 2\frac{\partial S^{\alpha\beta}}{\partial F_{\mu\nu}} - 2\frac{\partial N^{\alpha\beta}}{\partial F_{\mu\nu}} \tag{16}$$

and a similar expression with the pairs $\alpha\beta$ and $\mu\nu$ interchanged. By antisymmetrizing in these pairs one gets

$$B^{[\alpha\beta][\mu\nu]} = -2\frac{\partial N^{\alpha\beta}}{\partial F_{\mu\nu}} = -2\frac{\partial N^{\mu\nu}}{\partial F_{\alpha\beta}}\,. \tag{17}$$

On the other hand the symmetries (14) mean that $B^{\alpha\beta\mu\nu} = B^{[\alpha\beta][\mu\nu]}$ and by equating Eq. (16) to (17) one arrives at the following restrictions imposed on $d^{\alpha\beta}$,

$$\frac{\partial S^{\alpha\beta}}{\partial A_{(\mu;\nu)}} = \frac{\partial S^{\alpha\beta}}{\partial F_{\mu\nu}} = \frac{\partial N^{\alpha\beta}}{\partial A_{(\mu;\nu)}} = 0\,. \tag{18}$$

From these one infers that the Lagrangian may depend on $A_{(\alpha;\beta)}$ only via linear terms,

$$L = f_1(s)A^\alpha A^\beta A_{(\alpha;\beta)} + f_2(s)g^{\alpha\beta}A_{(\alpha;\beta)} + L_2(A_\mu, F_{\mu\nu})\,, \tag{19}$$

where f_1 and f_2 are arbitrary smooth functions of the vector length, $s \equiv A^\mu A_\mu$. The first two terms in (19) can be expressed as $\nabla_\alpha(f(s)A^\alpha)+h(s)A^\alpha_{;\alpha}$, with $2df/ds \equiv f_1$ and $h \equiv f_2 - f$. Then discarding the divergence term from the Lagrangian one finds

$$d^{\alpha\beta} = hg^{\alpha\beta} + N^{\alpha\beta} \qquad \text{and} \qquad B^{\alpha\beta\mu\nu} = -2\frac{\partial N^{\alpha\beta}}{\partial F_{\mu\nu}} = -2\frac{\partial N^{\mu\nu}}{\partial F_{\alpha\beta}}\,. \tag{20}$$

We next study the identities involving $A_{\nu;(\beta\gamma)}$ quadratically. From (13) one sees that vanishing of the coefficients of these terms requires

$$\frac{\partial B^{\mu(\beta\gamma)\nu}}{\partial F_{\sigma\lambda}} + \frac{\partial B^{\sigma(\beta\gamma)\nu}}{\partial F_{\mu\lambda}} \equiv 0\,. \tag{21}$$

By employing Eqs. (20), however, one cannot simplify these identities (after inserting (20) into (21) one recovers the identities with the pairs of indices $\beta\gamma$ and $\mu\sigma$ interchanged), therefore one passes to studying the identities involving $A_{\lambda;(\mu\sigma)}$ linearly. These imply vanishing of the curly bracket in Eq. (13). Making use of (20) this reads

$$F_{\alpha\nu}B^{\nu(\mu\sigma)\lambda} + A_\alpha C^{\mu\sigma\lambda} \equiv 0 \tag{22}$$

with

$$C^{\mu\sigma\lambda} \equiv 2h'(g^{\mu\sigma}A^\lambda - A^{(\mu}g^{\sigma)\lambda}) - \frac{1}{2}\left(\frac{\partial N^{\lambda\sigma}}{\partial A_\mu} + \frac{\partial N^{\lambda\mu}}{\partial A_\sigma}\right) = C^{(\mu\sigma)\lambda} \tag{23}$$

and $h' = dh/ds$. All the time one investigates a generic field A_μ for which $\det(F_{\mu\nu}) \neq 0$. (It is here that the even number of spacetime dimensions becomes relevant; for d odd, $\det(F_{\mu\nu}) = 0$.) Then there exists the inverse matrix $f^{\alpha\beta} = -f^{\beta\alpha}$ given by $f^{\alpha\mu}F_{\mu\beta} = F_{\beta\mu}f^{\mu\alpha} = \delta^\alpha_\beta$. Multiplying (22) by $f^{\alpha\tau}$ one gets

$$B^{\alpha(\mu\nu)\beta} = -f^{\alpha\sigma}A_\sigma C^{\mu\nu\beta}. \tag{24}$$

One immediately sees that this expression implies $A_\alpha B^{\alpha(\mu\nu)\beta} \equiv 0$. On the other hand one infers from (20) that $B^{\alpha(\mu\nu)\beta} \equiv B^{\beta(\mu\nu)\alpha}$. Applying this symmetry to the former identity one finds

$$B^{\alpha(\mu\nu)\beta}A_\beta = -f^{\alpha\sigma}A_\sigma C^{\mu\nu\beta}A_\beta \equiv 0\,. \tag{25}$$

This in turn implies that $C^{\mu\nu\beta}A_\beta \equiv 0$ since for a generic vector field $f^{\alpha\sigma}A_\sigma \neq 0$. By inspection of the expression (23) one concludes that the term proportional to h' in the latter identity contains no derivatives, whereas the other term in this identity is a sum of terms each of which does contain $F_{\mu\nu}$. In fact, terms of the form $\partial N^{\mu\nu}/\partial A_\alpha$ cannot involve an additive term free of the derivatives, since the latter would only arise from a linear term in the Lagrangian, $k^{\mu\nu}(A)F_{\mu\nu}$. However, for d even, an antisymmetric $k^{\mu\nu}$ cannot be made up alone of A_μ, $g_{\mu\nu}$ and the Levi–Civita tensor. In conclusion, the identity $C^{\mu\nu\beta}A_\beta \equiv 0$ splits into two sets,

$$h'(g^{\mu\nu}s - A^{(\mu}A^{\nu)}) \;\equiv\; 0 \tag{26}$$

and

$$A_\beta\left(\frac{\partial N^{\beta\nu}}{\partial A_\mu} + \frac{\partial N^{\beta\mu}}{\partial A_\nu}\right) \;\equiv\; 0\,. \tag{27}$$

Eq. (26) may be satisfied only if $h = h_0 = \text{const}$; then $h_0 A^\alpha_{;\alpha}$ is discarded as being a full divergence and finally $L = L(A_\mu, F_{\mu\nu}, g_{\mu\nu})$.

We now return to investigating the expression (24). It is very peculiar. First, the r.h.s. of the expression does not have the above mentioned symmetry $B^{\alpha(\mu\nu)\beta} \equiv B^{\beta(\mu\nu)\alpha}$. Second, according to the assumption of the theorem, $B^{\alpha\mu\nu\beta}$ is analytic (is a Taylor series) in $F_{\mu\nu}$ about $F_{\mu\nu} = 0$ and the same holds for $C^{\mu\nu\beta}$. Yet $B^{\alpha(\mu\nu)\beta}$ is proportional to $f^{\alpha\sigma} A_\sigma$ and the expression should be valid also about $F_{\mu\nu} = 0$ where the inverse $f^{\alpha\sigma}$ does not exist. (Actually the expression breaks down whenever $\det(F_{\mu\nu}) = 0$.) The dependence on $f^{\alpha\sigma}$ cannot be eliminated since there is no contraction of this tensor with any of $F_{\mu\nu}$ appearing in $C^{\mu\nu\beta}$. One then infers that identity (24) may hold only if $B^{\alpha(\mu\nu)\beta} \equiv 0$.

To avoid any confusion it should be emphasized that the above reasoning does not apply to the Noether identity (8). In fact, with the aid of $f^{\alpha\mu}$ it can be reexpressed as

$$E^\alpha = f^{\alpha\mu}(A_\mu E^\nu_{;\nu} + T_{\mu\nu}{}^{;\nu})$$

and apparently E^α does explicitly depend on $f^{\alpha\mu}$, contrary to the assumption of the theorem. However, here $f^{\alpha\mu}$ is contracted with $T_{\mu\nu}{}^{;\nu}$ and the original form of the identity ensures that the $f^{\alpha\mu}$-dependence is trivially cancelled. This fact stresses the role played by $T_{\mu\nu}$.

The identity $B^{\alpha(\mu\nu)\beta} \equiv 0$ makes identities (21) trivial and has two further consequences. First, together with (14) and (20) it implies that $B^{\alpha\mu\nu\beta}$ is totally antisymmetric,

$$B^{\alpha\mu\nu\beta} = B^{[\alpha\mu\nu\beta]}\,. \tag{28}$$

In $d = 4$ one then infers that $B^{\alpha\mu\nu\beta} = p(A_\lambda, F_{\lambda\sigma})\varepsilon^{\alpha\mu\nu\beta}$ with some definite pseudoscalar function p. Second, it requires $C^{\mu\nu\beta} \equiv 0$ or from (23),

$$\frac{\partial N^{\alpha\mu}}{\partial A_\nu} + \frac{\partial N^{\alpha\nu}}{\partial A_\mu} \equiv 0\,. \tag{29}$$

The latter shows that the following tensor is totally antisymmetric,

$$n^{\alpha\mu\nu} \equiv \frac{\partial N^{\alpha\mu}}{\partial A_\nu} = n^{[\alpha\mu\nu]}\,. \tag{30}$$

The formula (12) for the Euler operator is reduced, upon applying Eqs. (28) and (30), to

$$E^\mu = -\frac{1}{2} n^{\mu\alpha\beta} F_{\alpha\beta} + l^\mu, \tag{31}$$

now it contains no second order derivatives.

Finally we study the last system of identities arising from (13), those involving at most the first derivatives of A_μ, *i.e.* $a_\alpha \equiv 0$. After some manipulations with the use of the Riemann tensor symmetries and Eqs. (28) and (30) one arrives at the following expression:

$$a_\alpha = F_{\alpha\mu} E^\mu - A_\alpha \frac{\partial E^\mu}{\partial A_\nu} A_{\nu;\mu} \equiv 0 \tag{32}$$

with E^μ given by (31). As in the case of $B^{\alpha(\mu\nu)\beta}$ one may use the analyticity property to prove that $E^\mu \equiv 0$. It is interesting, however, to see that this result can also be attained in an independent way. Using the formula (31) one easily finds that the scalar in the last term of (32) is equal to

$$\frac{\partial E^\mu}{\partial A_\nu} A_{\nu;\mu} = \frac{\partial^2 L}{\partial A_\mu \partial A_\nu} A_{\mu;\nu} \,. \tag{33}$$

One sees that the r.h.s. of Eq. (33) depends linearly on $A_{(\mu;\nu)}$ and upon inserting the scalar back into Eq. (32) the term $F_{\alpha\mu} E^\mu$ acquires the same dependence. On the other hand it has already been proved that $L = L(A_\mu, F_{\mu\nu}, g_{\mu\nu})$ and the symmetrized derivative cannot arise in the process of differentiation of the Lagrangian. The contradiction is removed by requiring that

$$\frac{\partial^2 L}{\partial A_\mu \partial A_\nu} \equiv 0 \tag{34}$$

or $L = L^\mu(F_{\alpha\beta}) A_\mu + L_0(F_{\alpha\beta})$. However for even number of dimensions it is impossible to make up a vector out of $F_{\alpha\beta}$, $g_{\mu\nu}$ and the Levi–Civita tensor. In consequence $L^\mu = 0$ and the Lagrangian is some function of $F_{\alpha\beta}$ and $g_{\mu\nu}$ alone.

As a result of vanishing of the scalar (33) one gets from (32) that $F_{\alpha\mu} E^\mu \equiv 0$ and finally one arrives at the conclusion that for a generic vector field the Euler operator associated with the Lagrangian generating the stress tensor with $T_{\mu\nu}{}^{;\nu} \equiv 0$ vanishes identically, $E^\mu[L(A)] \equiv 0$. This outcome is in agreement with Eq. (31) since the Lagrangian does not depend on A_μ and then $n^{\alpha\mu\nu} \equiv 0 \equiv l^\mu$.

The last step of the proof consists in finding a relationship between the identity $T_{\mu\nu}{}^{;\nu} \equiv 0$ and the stress tensor itself in the case $d = 4$. From Eq. (28) one infers that

$$B^{\alpha\mu\nu\beta} = 4 \frac{\partial^2 L}{\partial F_{\alpha\mu} \partial F_{\nu\beta}} = p(F_{\lambda\sigma}) \varepsilon^{\alpha\mu\nu\beta} \tag{35}$$

with unknown pseudoscalar p. The Lagrangian, which depends on A_α only via $F_{\alpha\beta}$, is a function of the two invariants of the field strength, $V = F_{\alpha\beta} F^{\alpha\beta}$

and $W = P^2$ where $P = \varepsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu}$ is a pseudoscalar, *i.e.* $L = L(W, V)$. Then

$$\begin{aligned} B^{\alpha\mu\nu\beta} &= 32PL_{WV}(\varepsilon^{\alpha\mu\lambda\sigma}F^{\nu\beta} + F^{\alpha\mu}\varepsilon^{\nu\beta\lambda\sigma})F_{\lambda\sigma} \\ &\quad +32(L_W + 2WL_{WW})\varepsilon^{\alpha\mu\lambda\sigma}F_{\lambda\sigma}\varepsilon^{\nu\beta\tau\rho}F_{\tau\rho} \\ &\quad +16L_{VV}F^{\alpha\mu}F^{\nu\beta} + 4L_V(g^{\alpha\nu}g^{\mu\beta} - g^{\alpha\beta}g^{\mu\nu}) + 16PL_W\varepsilon^{\alpha\mu\nu\beta} \\ &\equiv J^{\alpha\mu\nu\beta} + 16PL_W\varepsilon^{\alpha\mu\nu\beta} = p(W,V)\varepsilon^{\alpha\mu\nu\beta}\,, \end{aligned} \tag{36}$$

where $L_V = \partial L/\partial V$ *etc.* The first four terms have lower symmetry than that required: only $\alpha\mu\nu\beta = [\alpha\mu][\nu\beta] = \nu\beta\alpha\mu$, thus their sum $J^{\alpha\mu\nu\beta}$ must vanish identically and this is possible only if each of them is separately zero. To see this one assumes that the sum is zero for a fixed field $F_{0\mu\nu}$. Let $F_{\mu\nu} = F_{0\mu\nu} + \delta F_{\mu\nu}$ where $\delta F_{\mu\nu}$ is arbitrary infinitesimal. The variation $\delta J^{\alpha\mu\nu\beta}$ also must vanish,

$$\begin{aligned} \delta J^{\alpha\mu\nu\beta} &= 32\delta(PL_{WV})(\varepsilon^{\alpha\mu\lambda\sigma}F_0^{\nu\beta} + F_0^{\alpha\mu}\varepsilon^{\nu\beta\lambda\sigma})F_{0\lambda\sigma} \\ &\quad +32PL_{WV}[(\varepsilon^{\alpha\mu\lambda\sigma}\delta F^{\nu\beta} + \delta F^{\alpha\mu}\varepsilon^{\nu\beta\lambda\sigma})F_{0\lambda\sigma} \\ &\quad +(\varepsilon^{\alpha\mu\lambda\sigma}F_0^{\nu\beta} + F_0^{\alpha\mu}\varepsilon^{\nu\beta\lambda\sigma})\delta F_{\lambda\sigma}] \\ &\quad +32\delta(L_W + 2WL_{WW})\varepsilon^{\alpha\mu\lambda\sigma}F_{0\lambda\sigma}\varepsilon^{\nu\beta\tau\rho}F_{0\tau\rho} \\ &\quad +32(L_W + 2WL_{WW})\varepsilon^{\alpha\mu\lambda\sigma}\varepsilon^{\nu\beta\tau\rho}(\delta F_{\lambda\sigma}F_{0\tau\rho} + F_{0\lambda\sigma}\delta F_{\tau\rho}) \\ &\quad +16\delta L_{VV}F_0^{\alpha\mu}F_0^{\nu\beta} + 16L_{VV}(\delta F^{\alpha\mu}F_0^{\nu\beta} + F_0^{\alpha\mu}\delta F^{\nu\beta}) \\ &\quad +4\delta L_V(g^{\alpha\nu}g^{\mu\beta} - g^{\alpha\beta}g^{\mu\nu}) \equiv 0\,. \end{aligned} \tag{37}$$

Here for any $\Phi(W,V)$ one has as usual $\delta\Phi = \Phi_W\delta W + \Phi_V\delta V$ where $\delta W = 2P_0\delta P = 4P_0\varepsilon^{\alpha\beta\mu\nu}F_{0\alpha\beta}\delta F_{\mu\nu}$ and $\delta V = 2F_0^{\mu\nu}\delta F_{\mu\nu}$, all the derivatives and W and V are taken at $F_{0\mu\nu}$. At fixed $F_{0\mu\nu}$ one varies $\delta F_{\mu\nu}$ in such a way that the variations δW and δV remain unaltered. Then the first, third, fifth and seventh (the last) terms in (37) remain constant whereas the other three are variable. For the identity holds for any $\delta F_{\mu\nu}$, the constant terms must vanish, *i.e.* $\delta(PL_{WV}) = \delta(L_W + 2WL_{WW}) = \delta L_{VV} = \delta L_V = 0$ and these identities hold for any values of δW and δV. The last identity yields then $L_{VW} = L_{VV} = 0$ or $L(W,V) = L(W) + aV$, a is constant. Inserting this Lagrangian into Eq. (36) one sees that the first and third terms are zero whereas the fourth term becomes independent of $F_{\mu\nu}$, thus it must vanish too. This implies $L_V = 0$ or $a = 0$. The vanishing sum $J^{\alpha\mu\nu\beta} = 0$ consists now of the second term alone and it is zero provided

$L_W + 2WL_{WW} = 0$. A general solution to this equation is (dropping an additive constant) $L = c\sqrt{W} = c|P|$ with constant c. Finally,

$$L = 2c|\nabla_\alpha(\varepsilon^{\alpha\beta\mu\nu}A_\beta F_{\mu\nu})|,$$

a textbook result [7]. This completes the proof of Proposition 2.

For dimensions $d = 2n$, $n > 2$, the tensor $F_{\alpha\beta}$ has more invariants. One may conjecture that also in higher dimensions Eq. (28) implies that

$$L = c|\varepsilon^{\alpha_1\beta_1\ldots\alpha_n\beta_n}F_{\alpha_1\beta_1}\ldots F_{\alpha_n\beta_n}|$$

being a full divergence and hence $T_{\mu\nu}(A) \equiv 0$. However, it is harder to prove that this solution is unique.

In principle one might envisage a scalar (or $n < d$ scalars) or a single vector field in $d > 4$ even with $T_{\mu\nu} \neq 0$ and $T^{\mu\nu}{}_{;\nu} \equiv 0$. Formally such a field might appear in Einstein field equations. However, since the Lagrange equations of motion are trivial and the field has no determined propagation, it should be rejected on physical grounds. At first sight the stress tensor for a ground state solution of a quantum field, $\langle T_{\mu\nu}\rangle = \rho_V g_{\mu\nu}$, where ρ_V is a constant energy density for the classical nonzero value of the quantum field in this state, contradicts the above statement. It should be therefore emphasized that this expectation value of $T_{\mu\nu}$ comes from semiclassical considerations and cannot be derived within classical field theory (formally this expression is generated by the cosmological constant term in the full Lagrangian including gravity); the proposition does not apply to that case.

The Proposition 2 cannot be generalized either to odd number of dimensions or to more than one vector field. This is shown by the following counterexamples.

Example 2

Let $d = 3$. If one chooses a pseudoscalar $L = \varepsilon^{\alpha\beta\gamma}F_{\alpha\beta}A_\gamma$, then $T_{\mu\nu}(A) \equiv 0$. This Lagrangian is not a divergence and $E^\mu = 2\varepsilon^{\mu\alpha\beta}F_{\alpha\beta}$, the operator involves no second order derivatives since L is degenerate being linear in $F_{\alpha\beta}$. The field equations read then $F_{\alpha\beta} = 0$ and admit only one solution, the vacuum. The model is trivial.

Example 3

In $d = 5$ the analogous model is nontrivial. Let

$$L = W^\mu A_\mu = \varepsilon^{\mu\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}A_\mu\,, \tag{38}$$

where $W^\mu \equiv \varepsilon^{\mu\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}$. The stress tensor is zero, $T_{\mu\nu}(A) \equiv 0$, whereas the Lagrangian is not a divergence and is gauge invariant under $A'_\mu = A_\mu + \partial_\mu\chi$. L is apparently nondegenerate as being quadratic in $F_{\alpha\beta}$, nevertheless the Euler operator is of the first order, $E^\mu = 3W^\mu$, and in this sense the propagation equations for the field are degenerate. One finds that $E^{\mu}_{;\mu} \equiv 0$ owing to the gauge invariance. Then the Noether identity (8) reduces to $F_{\alpha\mu}E^{\mu} \equiv 0$ with E^{μ} nonvanishing in general ($\det(F_{\mu\nu}) \equiv 0$). The field equations $W^{\mu} = 0$ are quadratic in $F_{\alpha\beta}$ and admit nonzero solutions. A particular solution is, *e.g.*

$$A_1 = a_1 x^0 - a_2 x^2 - a_3 x^3 - a_4 x^4, \qquad A_0 = A_2 = A_3 = A_4 = 0\,, \tag{39}$$

yielding $F_{01} = a_1$, $F_{12} = a_2$, $F_{13} = a_3$ and $F_{14} = a_4$, otherwise zero, with $a_1, \ldots, a_4$ constant.

Example 4

Next we consider two vector fields forming an open system, $d = 4$. A field A_μ interacts with a given external field W^{μ}. First one defines an antisymmetric tensor $V_{\mu\nu} \equiv W^{\alpha}A_{[\alpha}\partial_{\mu}A_{\nu]} = \frac{1}{2}W^{\alpha}A_{[\alpha}F_{\mu\nu]}$, where as usual, $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$. The tensor is metric independent and thus may be used to constructing an interaction Lagrangian,

$$\begin{aligned} L(A,W) &\equiv \frac{1}{\sqrt{-g}}[-\det(V_{\mu\nu})]^{1/2} = \frac{1}{\sqrt{8}}[2V_{\alpha\beta}V^{\beta\mu}V_{\mu\nu}V^{\nu\alpha} - (V_{\mu\nu}V^{\mu\nu})^2]^{1/2} \\ &= \frac{1}{8}|\varepsilon^{\alpha\beta\mu\nu}V_{\alpha\beta}V_{\mu\nu}|\,. \end{aligned} \tag{40}$$

L does not involve derivatives of the external field. In this model there is no free Lagrangian for A_μ and obviously $L(A,0) \equiv 0$. The stress tensor generated by the interaction Lagrangian is $T_{\mu\nu}(A,W) \equiv 0$. On the other hand if one attempts to complete the system, *i.e.* to make it closed, by adding a Lagrangian L_W for W^{μ} (free or including an interaction term), then $\sqrt{-g}L_W$ must depend on the metric. Thus a full stress tensor for a closed system of the two vector fields is different from zero.

The open system described by L as in (40) has no simple symmetries. The Euler operator E^{μ} for A_μ is nondegenerate and very complicated. The second order derivatives appear in it in a term of the form

$$\varepsilon^{\mu\nu\alpha\beta}A_{\nu}A_{\alpha;(\beta\gamma)}W^{\gamma}W^{\sigma}A_{\sigma}. \tag{41}$$

Example 5

One can construct a model for a closed system of two interacting vector fields in $d = 4$. One assigns field strengths to vector potentials A_μ and B_μ:

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \qquad \text{and} \qquad H_{\mu\nu} = B_{\nu,\mu} - B_{\mu,\nu}\,. \tag{42}$$

The interaction Lagrangian is chosen as

$$L(A,B) \equiv \frac{1}{\sqrt{-g}}\,[\det(F_{\mu\nu})\det(H_{\mu\nu})]^{1/4} = \frac{1}{8}|L_A(F)L_B(H)|^{1/2}\,, \tag{43}$$

where $L_A \equiv \varepsilon^{\alpha\beta\mu\nu} F_{\alpha\beta} F_{\mu\nu}$ and $L_B \equiv \varepsilon^{\alpha\beta\mu\nu} H_{\alpha\beta} H_{\mu\nu}$ are pseudoscalars. As in Example 4 there are no free Lagrangians for the fields (by introducing free Lagrangians one will render the full Lagrangian density metric dependent, Proposition 2) and the interaction Lagrangian is gauge invariant under $A'_\mu = A_\mu + \partial_\mu \chi_1$ and $B'_\mu = B_\mu + \partial_\mu \chi_2$, χ_1 and χ_2 arbitrary. The interaction energy and momentum for the system are zero, $T_{\mu\nu}(A, B) \equiv 0$, whereas the Euler operator consists of two vector ones,

$$E^\mu_A \equiv \frac{\delta L}{\delta A_\mu} = -\frac{L}{L_A} \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} \left(\frac{L_{A;\nu}}{L_A} - \frac{L_{B;\nu}}{L_B} \right) \tag{44}$$

and analogously, by applying the symmetry of interchanging the fields in L,

$$E^\mu_B \equiv \frac{\delta L}{\delta B_\mu} = -\frac{L}{L_B} \varepsilon^{\mu\nu\alpha\beta} H_{\alpha\beta} \left(\frac{L_{B;\nu}}{L_B} - \frac{L_{A;\nu}}{L_A} \right). \tag{45}$$

The operators contain second order derivatives (are nondegenerate). One can find a special solution to the field equations $E^\mu_A = 0 = E^\mu_B$ in Minkowski spacetime in the following way. One assumes that the "electric" and "magnetic" fields corresponding to $F_{\mu\nu}$ are parallel and of equal length and direction, then $F^{\mu\nu} F_{\mu\nu} = 0$ and $L_A = -8a^2$, $a > 0$ with $F_{01} = F_{32} = a$, otherwise zero; and analogously, $H^{\mu\nu} H_{\mu\nu} = 0$ and $L_B = -8b^2$, $b > 0$ and $H_{01} = H_{32} = b$. In this case the two operators are proportional,

$$E^\mu_A = \frac{b}{2} (-\partial_1, \partial_0, \partial_3, -\partial_2) \ln \frac{a}{b} = -\frac{a}{b} E^\mu_B \,. \tag{46}$$

A particular solution is then $A_\mu = -a(x, 0, 0, y)$ and $B_\mu = -b(x, 0, 0, y)$ with a and b constant. Eq. (46) shows that a and b cannot vanish (no free fields).

3. The stress tensor vanishing for solutions

The Propositions 1 and 2 raise an obvious problem: is it possible that $T_{\mu\nu}$, while being different from zero for some values ("off shell") of the field under consideration, ψ_A, vanishes for all solutions of $E^A = 0$ and not only for the trivial solution $\psi_A = 0$? Such a possibility would be more disturbing than the case $T_{\mu\nu} \equiv 0$ for all values of the field, for field configurations which do not obey some equations of motion do not exist in nature. Such a matter, subject to deterministic causal propagation equations, would be a truly nongravitating one. In the framework of classical field theory one can almost exclude such fields under reasonable assumptions. We again consider only scalar fields and a single vector field.

Proposition 3. For a system of arbitrary number of interacting scalar fields minimally coupled to gravity and described by nontrivial Lagrange equations of motion, the full stress tensor cannot vanish for all solutions if it is not identically zero.

Proof of Proposition 3. For minimally coupled scalars any Lagrangian is of the form $L(\phi_a, \phi_{a,\mu}, g_{\mu\nu})$ and is free of the metric connection and this results in the stress tensor free of the second order covariant derivatives, *i.e.* $T_{\mu\nu} = T_{\mu\nu}(\phi_a, \phi_{a,\mu}, g_{\mu\nu})$. This means that on a Cauchy surface the stress $T_{\mu\nu}$ is determined solely by the initial data and thus can be given (since it does not vanish identically) any prescribed value (apart from some restrictions) independently of the field equations. By continuity, the stress tensor is also different from zero in some neighbourhood of the Cauchy surface.

In the case of nonminimal coupling the proposition does not apply directly. However, all known cases comprise the scalar field of scalar–tensor gravity theories (all generalizations of Brans–Dicke theory), the conformally invariant scalar field (which mathematically is merely a special case of scalar–tensor gravity) and a scalar field arising in restricted metric nonlinear gravity theories (Lagrangian being a smooth function of the curvature scalar) via a suitable Legendre transformation. The nonminimal coupling takes the form $f(\phi)R$, the Lagrangian may or may not contain a kinetic term for ϕ. Then one introduces a new spacetime metric $\tilde{g}_{\mu\nu}$ by means of a Legendre map (actually this map is a conformal map of the original metric $g_{\mu\nu}$) and suitably redefines the scalar, $\tilde{\phi} = \tilde{\phi}(\phi)$. The mapping is commonly denoted as a transition from Jordan frame, *i.e.* the system $(g_{\mu\nu}, \phi)$, to Einstein frame consisting of $\tilde{g}_{\mu\nu}$ and $\tilde{\phi}$. In Einstein frame $\tilde{\phi}$ is minimally coupled to $\tilde{g}_{\mu\nu}$ (which should be regarded as the physical spacetime metric) [8, 9] and Proposition 3 works.

Proposition 4. For a single vector field A_μ in Minkowski spacetime there exist solutions in the space of all solutions vanishing sufficiently quickly at spatial infinity for which $T_{\mu\nu}(A) \neq 0$.

Proof of Proposition 4. A first order Lagrangian $L(A_\mu, A_{\mu,\nu})$ in Cartesian coordinates generates $T_{\mu\nu}(A_\alpha, A_{\alpha,\beta}, A_{\alpha,\beta\gamma})$ explicitly depending on second derivatives except the case where L depends on $A_{\mu,\nu}$ only via $F_{\mu\nu}$. Let S be a Cauchy surface with a unit timelike normal vector n^ν, $n^\nu n_\nu = -1$. The initial data for A_μ form a set $C_A \equiv \{A_\mu, n^\nu A_{\mu,\nu}\}$ of functions given on S. On the boundary 2-sphere ∂S at spatial infinity the initial data vanish.

One integrates the Belinfante–Rosenfeld identity (Eq. (A.12) in Appendix), expressed in Cartesian coordinates, over S,

$$0 \equiv \int_S (T^{\mu\nu} - t^{\mu\nu} + A^\mu E^\nu + \partial_\alpha K^{\mu\nu\alpha}) n_\nu \, dS\,. \tag{47}$$

On S the tensor $K^{\mu\nu\alpha}$ being a linear combination of the classical spin tensor $S^{\mu\nu\alpha}$, Eqs. (A.8)–(A.9), is determined by the initial data, $K^{\mu\nu\alpha} = K^{\mu\nu\alpha}(C_A)$. Then applying the antisymmetry $K^{\mu\nu\alpha} = -K^{\mu\alpha\nu}$ and the Gauss' formula one can replace the integral of $n_\nu \partial_\alpha K^{\mu\nu\alpha}$ over S by the integral of $K^{\mu\nu\alpha}$ over ∂S and the latter is zero. Let A_μ be a solution of $E^\nu = 0$ corresponding to the initial data C_A. Eq. (47) reduces to

$$\int\limits_S (T^{\mu\nu} - t^{\mu\nu}) n_\nu \, dS = 0 \,. \tag{48}$$

The canonical energy-momentum tensor $t^{\mu\nu}$, Eq. (A.7), is also determined on S by the initial data, whereas the stress tensor is determined by the solution A_μ. Since $t^{\mu\nu}$ is not zero in general, one can choose such initial data C_A that the integral of $t^{\mu\nu}(C_A)$ does not vanish. Then the conserved total 4-momentum of the field for the solution is different from zero,

$$P^\mu = \int\limits_S T^{\mu\nu}(A) n_\nu \, dS = \int\limits_S t^{\mu\nu}(C_A) n_\nu \, dS \neq 0 \,. \tag{49}$$

The Proposition is proved.

In this form the proof works only in Minkowski spacetime. Let $A_{0\mu}$ be a solution in flat spacetime for which Eq. (49) holds. Making a small perturbation $A_\mu = A_{0\mu} + \epsilon_\mu$ and $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ (where $h_{\mu\nu}$ may be a solution to Einstein field equations with some matter source, not necessarily equal to A_μ) one finds that $T_{\mu\nu}$ is altered by a small quantity $T^L_{\mu\nu}$ which is linear in ϵ_μ and $h_{\mu\nu}$. One then concludes that for spacetimes which are sufficiently close to flat space and for solutions being a small perturbation of $A_{0\mu}$, the stress tensor is nonzero too.

Furthermore, the proof of the Proposition may be generalized to at least one class of curved spacetimes. Let a spacetime metric admit a covariantly constant vector field k_μ, $k_{\mu;\nu} = 0$. For example, if the spacetime is empty, $R_{\mu\nu} = 0$ (and $d = 4$), then k_μ is null and the spacetime represents the plane-fronted gravitational wave. In such a spacetime one considers a Cauchy surface extending to the spatial infinity. As previously, the initial data for A_μ are $C_A = \{A_\mu, n^\nu A_{\mu;\nu}\}$ with n^ν the unit normal vector to S; the data vanish on the boundary ∂S at spatial infinity. The tensors $K^{\mu\nu\alpha}$ and $t_{\mu\nu}$ are determined on S by the initial data whereas $T_{\mu\nu}$ can be evaluated only for the solution corresponding to C_A.

One multiplies the identity (A.12) by k_μ and integrates it over S for a solution. The integral of the divergence term, $k_\mu n_\nu \nabla_\alpha K^{\mu\nu\alpha} = n_\nu \nabla_\alpha (k_\mu K^{\mu[\nu\alpha]})$, is equal to the integral over ∂S and hence is zero. One then arrives at an expression analogous to (48),

$$\int_S k_\mu (T^{\mu\nu} - t^{\mu\nu}) n_\nu \, dS = 0 \,. \tag{50}$$

Once again, choosing such initial data that the integral of $k_\mu t^{\mu\nu} n_\nu$ is nonzero, one gets $T^{\mu\nu}(A) \neq 0$. Unfortunately, the class of spacetimes admitting a covariantly constant Killing field is rather narrow.

Finally we remark that Propositions 3 and 4 cannot be further strengthened, *i.e.* one cannot exclude the situation that the stress tensor vanishes in the whole spacetime for some particular solutions of the Lagrange equations of motion. In fact, at least one counterexample is known: a nonlinear massive spin-two field generated by a higher derivative gravity theory. For this field $T_{\mu\nu} = 0$ in the spacetime of a plane-fronted gravitational wave [10].

4. The canonical energy-momentum tensor in a curved spacetime

In many theoretical investigations in classical and quantum field theory in Minkowski spacetime one employs the canonical energy-momentum tensor due to its conceptual simplicity and "naturalness". Yet the variational stress tensor is constructed in a way which is quite artificial in flat spacetime and one must appeal to arguments from outside the Lagrange formalism to show that the latter rather than the former is the physical energy-momentum tensor [1, 4].

The canonical energy-momentum tensor for a field ψ_A is

$$t_\mu{}^\nu(\psi) \equiv \delta^\nu_\mu L - \psi_{A;\mu} \frac{\partial L}{\partial \psi_{A;\nu}} \,. \tag{51}$$

Taking divergence of the tensor and employing Ricci identity (the formula after Eq. (A.15)) and Eq. (A.14) one finds that

$$\nabla_\nu t_\mu{}^\nu(\psi) = E^A \psi_{A;\mu} + \frac{\partial L}{\partial \psi_{A;\nu}} Z_A{}^\beta{}_\alpha R^\alpha{}_{\beta\nu\mu} \,. \tag{52}$$

This means that even if the field equations hold, $E^A = 0$, the canonical tensor is not conserved in a curved spacetime. This is the case of the electromagnetic (Maxwell) field,

$$\nabla_\nu t_\mu{}^\nu(A) = \frac{1}{4\pi} A_\sigma F^{\alpha\beta} R^\sigma{}_{\alpha\beta\mu} \,. \tag{53}$$

The above and other bizarre properties of the canonical tensor are due to the fact that the tensor, as defined in Eq. (51), does not fit well to the variational formalism of field theory. In fact, any full divergence term in the field Lagrangian does not contribute to E^A nor to $T_{\mu\nu}(\psi)$, thus whether one discards such terms or not whilst evaluating these quantities does not affect the final outcome. Yet divergence terms in L do contribute to the canonical tensor. For example, let for a vector field $L = \nabla_\alpha(f(A^\mu A_\mu)A^\alpha)$, then $t^{\mu\nu}(A) = \nabla_\alpha(fg^{\mu\nu}A^\alpha - fg^{\mu\alpha}A^\nu)$.

This example illustrates a generic feature of $t^{\mu\nu}$ for vector fields: if the Lagrangian is a total divergence, then $T_{\mu\nu}(A) \equiv 0 \equiv E^\mu$ and the Belinfante–Rosenfeld identity (A.12) yields

$$t^{\mu\nu} = \frac{1}{2}\nabla_\alpha(S^{\mu\nu\alpha} + S^{\alpha\mu\nu} + S^{\alpha\nu\mu}) \tag{54}$$

with a nonvanishing spin tensor. This vector field has no physical propagation and carries no energy, yet $t^{\mu\nu}$ apparently attributes to it some nontrivial feature. This fact convincingly shows that the canonical energy-momentum tensor is not a physical quantity.

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Appendix

For the reader's convenience we rederive here the Belinfante–Rosenfeld identity for any vector field generalized to a curved spacetime since it cannot be easily found in the literature in the form suitable for the present paper. The identity relates the variational and the canonical energy-momentum tensors with the classical spin tensor and the Lagrange equations. Our approach is closest to that in [11–13], for a different approach see [14–16]. (The Belinfante–Rosenfeld identity for the vacuum electromagnetic field, *i.e.* Maxwell's equations hold, in a curved spacetime is implicitly given in [17].) We also recall the derivation of the Noether identity for any matter tensor field arising from the diffeomorphism invariance of the field action in the case of any metric theory of gravity. For an arbitrary vector matter field we show the explicit relationship between the generalized Belinfante–Rosenfeld identity and the Noether identity.

Let a classical vector matter field A_μ on a spacetime $(M, g_{\mu\nu})$ be described by a scalar Lagrangian L. The Lagrangian may be written either in the explicitly covariant way as a function of tensors or in the noncovariant form as a function of tensors and their partial derivatives,

$$\begin{aligned} L(A_\mu, A_{\mu;\nu}, g^{\alpha\beta}) &= L\left(A_\mu, A_{\mu,\nu} - \Gamma^\sigma_{\nu\mu} A_\sigma, g^{\alpha,\beta}\right) \\ &\equiv L'\left(A_\mu, A_{\mu,\nu}, g^{\alpha\beta}, g^{\alpha\beta}{}_{,\nu}\right) . \end{aligned} \tag{A.1}$$

The Lie derivative of the scalar density (of the weight +1) $\sqrt{-g}L$ with respect to an arbitrary vector field ξ^μ may be evaluated either with the aid of the formula

$$\begin{aligned} \mathcal{L}_\xi(\sqrt{-g}L) &= \sqrt{-g}\nabla_\mu(L\xi^\mu) \\ &= \sqrt{-g}\left[L\xi^\mu{}_{;\mu} + \xi^\mu\left(\frac{\partial L}{\partial A_\alpha}A_{\alpha;\mu} + \frac{\partial L}{\partial A_{\alpha;\beta}}A_{\alpha;\beta\mu}\right)\right], \end{aligned} \tag{A.2}$$

or by Lie differentiating it as a composite function,

$$\begin{aligned} \mathcal{L}_\xi(\sqrt{-g}L) &= \mathcal{L}_\xi(\sqrt{-g}L') = \frac{\partial(\sqrt{-g}L')}{\partial A_\mu}\mathcal{L}_\xi A_\mu + \frac{\partial(\sqrt{-g}L')}{\partial A_{\mu,\nu}}\mathcal{L}_\xi(\partial_\nu A_\mu) \\ &\quad + \frac{\partial(\sqrt{-g}L')}{\partial g^{\alpha\beta}}\mathcal{L}_\xi g^{\alpha\beta} + \frac{\partial(\sqrt{-g}L')}{\partial g^{\alpha\beta}{}_{,\nu}}\mathcal{L}_\xi g^{\alpha\beta}{}_{,\nu} \\ &= \left[\frac{\partial(\sqrt{-g}L')}{\partial A_\mu} - \partial_\nu\left(\frac{\partial(\sqrt{-g}L')}{\partial A_{\mu,\nu}}\right)\right]\mathcal{L}_\xi A_\mu \\ &\quad + \left[\frac{\partial(\sqrt{-g}L')}{\partial g^{\alpha\beta}} - \partial_\nu\left(\frac{\partial(\sqrt{-g}L')}{\partial g^{\alpha\beta}{}_{,\nu}}\right)\right]\mathcal{L}_\xi g^{\alpha\beta} \\ &\quad + \partial_\nu\left[\frac{\partial(\sqrt{-g}L')}{\partial A_{\mu,\nu}}\mathcal{L}_\xi A_\mu + \frac{\partial(\sqrt{-g}L')}{\partial g^{\alpha\beta}{}_{,\nu}}\mathcal{L}_\xi g^{\alpha\beta}\right]. \end{aligned} \tag{A.3}$$

One uses the noncovariant form L' in order to arrive at the variational stress tensor and in the other differentiations one replaces L' by L. By applying $\mathcal{L}_\xi A_\mu = \xi^\nu A_{\mu;\nu} + A_\nu \xi^\nu{}_{;\mu}$ and $\mathcal{L}_\xi g^{\alpha\beta} = -2\xi^{(\alpha;\beta)}$ one recasts (A.3) to a form explicitly exhibiting the dependence of the Lie derivative on ξ^μ and its first and second order derivatives. First, denoting

$$d^{\alpha\beta} \equiv \frac{\partial L}{\partial A_{\alpha;\beta}}$$

and

$$E^\mu[L(A)] \equiv \frac{\partial L}{\partial A_\mu} - \nabla_\alpha\left(\frac{\partial L}{\partial A_{\mu;\alpha}}\right),$$

the Euler operator, one finds

$$\frac{\partial(\sqrt{-g}L')}{\partial A_\mu} = \sqrt{-g}\left(\frac{\partial L}{\partial A_\mu} + \frac{\partial L}{\partial A_{\alpha;\beta}}\frac{\partial A_{\alpha;\beta}}{\partial A_\mu}\right)$$
$$= \sqrt{-g}\left(\frac{\partial L}{\partial A_\mu} - d^{\alpha\beta}\Gamma^\mu_{\alpha\beta}\right)$$

and

$$\partial_\nu\left(\frac{\partial(\sqrt{-g}L')}{\partial A_{\mu,\nu}}\right) = \sqrt{-g}\left(d^{\mu\nu}{}_{;\nu} - d^{\alpha\beta}\Gamma^\mu_{\alpha\beta}\right).$$

Then the first square bracket in (A.3) equals $\sqrt{-g}E^\mu$. The second square bracket is just $-\frac{1}{2}\sqrt{-g}T_{\alpha\beta}(A)$. The first term in the third square bracket is also simple,

$$\frac{\partial(\sqrt{-g}L')}{\partial A_{\mu,\nu}} = \sqrt{-g}\frac{\partial L}{\partial A_{\mu;\nu}},$$

whereas the second term in this bracket is a complicated tensor density. After several manipulations with the last bracket and upon equating (A.2) to (A.3) one gets a scalar identity of the form

$$\xi^\mu P_\mu + \xi_{\mu;\nu}Q^{\mu\nu} + \xi_{\alpha;(\mu\nu)}R^{\alpha(\mu\nu)} \equiv 0\,. \tag{A.4}$$

The identity holds for any vector fields A_μ and ξ^μ. At any spacetime point the tensors ξ_μ, $\xi_{\mu;\nu}$ and $\xi_{\alpha;(\mu\nu)}$ are independent quantities and therefore their coefficients must identically vanish. Thus the identity splits into a cascade of 4+16+40 identities. The identities $R^{\alpha(\mu\nu)} \equiv 0$ are purely algebraic functions of of A_μ, $g_{\mu\nu}$ and $d^{\mu\nu}$ and are completely trivial in the sense that they hold for any values of these 3 tensors. The identities $P_\mu \equiv 0$ are also algebraic functions of these tensors and $R_{\alpha\beta\mu\nu}$ (arising from $\xi_{\alpha;[\mu\nu]}$) and are trivially satisfied provided $d^{\mu\nu}$ equals $\partial L/\partial A_{\mu;\nu}$. Only $Q^{\mu\nu} \equiv 0$ are nontrivial and read

$$Q^{\mu\nu} = T^{\mu\nu} + A^\mu E^\nu - Lg^{\mu\nu} + A_\alpha{}^{;\mu}d^{\alpha\nu} + \nabla_\alpha K^{\mu\nu\alpha} \equiv 0\,, \tag{A.5}$$

where

$$K^{\mu\nu\alpha} \equiv A^\mu N^{\nu\alpha} - S^{\mu\alpha}A^\nu + S^{\mu\nu}A^\alpha \tag{A.6}$$

and $S^{\mu\nu} \equiv d^{(\mu\nu)}$ and $N^{\mu\nu} \equiv d^{[\mu\nu]}$. One introduces the Pauli's canonical energy-momentum tensor

$$t_\mu{}^\nu(A) \equiv \delta^\nu_\mu L - A_{\alpha;\mu}\frac{\partial L}{\partial A_{\alpha;\nu}} \tag{A.7}$$

and splits $K^{\mu\nu\alpha}$ into its symmetric and antisymmetric parts w.r.t. to the first two indices,

$$K^{[\mu\nu]\alpha} = \tfrac{1}{2}(A^\mu d^{\nu\alpha} - A^\nu d^{\mu\alpha}) \equiv \tfrac{1}{2}S^{\mu\nu\alpha}\,, \tag{A.8}$$

$$K^{(\mu\nu)\alpha} = A^\alpha d^{(\mu\nu)} - d^{\alpha(\mu}A^{\nu)} = S^{\alpha(\mu\nu)}\,. \tag{A.9}$$

The tensor $S^{\mu\nu\alpha}$ may be interpreted as the density of the classical spin (helicity) of the vector field. In fact, for the electromagnetic field in vacuum in Minkowski spacetime the spin density tensor is defined as

$$S^{\mu\nu\alpha}_{\mathrm{EM}} = x^\mu(T^{\nu\alpha} - t^{\nu\alpha}) - x^\nu(T^{\mu\alpha} - t^{\mu\alpha})\,. \tag{A.10}$$

For $L_{\mathrm{EM}} = -1/(16\pi)F_{\alpha\beta}F^{\alpha\beta}$ and assuming that Maxwell equations hold, ${F^{\alpha\beta}}_{,\beta} = 0$, and the field vanishes sufficiently quickly at spatial infinity, one can remove a full divergence term from the spin density tensor since it does not contribute to the total spin of the electromagnetic field. As a result,

$$S^{\mu\nu\alpha}_{\mathrm{EM}} = \frac{1}{4\pi}(A^\mu F^{\nu\alpha} - A^\nu F^{\mu\alpha}) = A^\mu d^{\nu\alpha} - A^\nu d^{\mu\alpha}, \tag{A.11}$$

[11, 18]. The tensor $K^{\mu\nu\alpha}$ is antisymmetric, $K^{\mu\nu\alpha} = -K^{\mu\alpha\nu}$. Finally (A.5) takes on the form

$$Q^{\mu\nu} = T^{\mu\nu} - t^{\mu\nu} + A^\mu E^\nu + \frac{1}{2}\nabla_\alpha(S^{\mu\nu\alpha} + S^{\alpha\mu\nu} + S^{\alpha\nu\mu}) \equiv 0\,. \tag{A.12}$$

These are the Belinfante–Rosenfeld identities [19, 20] for any vector field, generalized to a curved spacetime. Notice that the identities arise due to the fact that the field Lagrangian is a scalar, hence the action is invariant under infinitesimal coordinate transformations (with appropriate boundary conditions).

The proof of Propositions 1 and 2 in Sec. 2 is based on the famous second Noether theorem (also named "the Noether identities", "the strong conservation laws" or "Bianchi identities for matter") [21–23]; our approach is based on [3, 23]. Let ψ_A denote an arbitrary tensor matter field (or a set of tensor fields) with a collective index A, described by a scalar Lagrangian $L(\psi) = L(\psi_A, \psi_{A;\mu}, g_{\mu\nu}, R_{\alpha\beta\mu\nu})$, *i.e.* one admits a possible non-minimal coupling to the curvature. The action integral for for Einstein gravity (actually one may envisage any metric theory of gravity since the derivation goes then without alteration) and the matter field is (we set $c = 8\pi G = 1$)

$$S = \int\limits_{\Omega} \left(\frac{1}{2}R + L\right)\sqrt{-g}\,d^4x\,. \tag{A.13}$$

The Euler operator for the equations of motion for ψ_A is

$$E^A[\psi, g] \equiv \frac{\partial L}{\partial \psi_A} - \nabla_\mu \left(\frac{\partial L}{\partial \psi_{A;\mu}} \right). \tag{A.14}$$

The action (A.13) is diffeomorphism invariant and in particular it remains unchanged by an infinitesimal point transformation $x'^\mu = x^\mu + \xi^\mu(x)$ with arbitrary ξ^μ which vanishes on the boundary of the domain Ω. The integration domain is thus mapped onto itself whereas the variations of the fields are given by Lie derivatives, $\delta\psi_A = -\mathcal{L}_\xi \psi_A$ and $\delta g^{\mu\nu} = -\mathcal{L}_\xi g^{\mu\nu}$. These variations can be expressed in terms of coefficients $Z_A{}^\beta{}_\alpha(\psi)$, being linear functions of ψ_A,

$$\mathcal{L}_\xi \psi_A = \xi^\alpha \psi_{A;\alpha} + Z_A{}^\beta{}_\alpha \xi^\alpha{}_{;\beta}\,, \tag{A.15}$$

which also appear in the covariant derivative,

$$\nabla_\mu \psi_A = \partial_\mu \psi_A - Z_A{}^\beta{}_\alpha(\psi) \Gamma^\alpha_{\mu\beta}$$

and in Ricci identity,

$$\psi_{A;\mu\nu} - \psi_{A;\nu\mu} = Z_A{}^\beta{}_\alpha(\psi) R^\alpha{}_{\beta\mu\nu}\,.$$

The invariance of the action implies

$$0 = \delta S = \int\limits_\Omega \left[\frac{1}{2}(G_{\mu\nu} - T_{\mu\nu})\delta g^{\mu\nu} + E^A \delta\psi_A \right] \sqrt{-g}\, d^4x \tag{A.16}$$

plus a surface integral which vanishes due to the boundary conditions. Applying (A.15), dropping again a total divergence and making use of the ordinary Bianchi identity $G^\nu_{\mu;\nu} \equiv 0$ one finds that the integrand is of the form $\xi^\mu B_\mu(\psi)$. Vanishing of the integral and arbitrariness of ξ^μ in the interior of Ω entail $B_\mu \equiv 0$ or

$$E^A \psi_{A;\alpha} \equiv \nabla_\beta (T_\alpha{}^\beta + E^A Z_A{}^\beta{}_\alpha)\,. \tag{A.17}$$

This is the Noether identity for any classical tensor matter field, valid for any dimension $d \geq 3$. Notice that the second Noether theorem derived in Refs. [12,24] does not include the variational (stress) energy-momentum tensor $T_{\mu\nu}(\psi)$. The tensor is absent there because it is (implicitly) assumed in these papers that the matter action is invariant under a symmetry transformation of the matter field (a gauge transformation) alone and the metric is left unaltered.

In the case of a vector field A_μ the coefficients are $Z_\mu{}^\beta{}_\alpha = \delta_\mu^\beta A_\alpha$ and the four Noether identities

$$E^\mu F_{\alpha\mu} \equiv T_{\alpha\beta}{}^{;\beta} + E^\mu{}_{;\mu} A_\alpha \,, \tag{A.18}$$

where $F_{\alpha\beta} \equiv A_{\beta;\alpha} - A_{\alpha;\beta}$, can be derived from the Belinfante–Rosenfeld identities (A.12). In fact, a direct and quite uphill calculation proves that

$$Q_{\alpha\mu}{}^{;\mu} = T_{\alpha\mu}{}^{;\mu} - F_{\alpha\mu}E^\mu + A_\alpha E^\mu{}_{;\mu} \equiv 0\,. \tag{A.19}$$

Analogous relationships exist for tensor fields.

REFERENCES

[1] S.W. Hawking, G.F.R. Ellis, *The Large Scale Structure of Space–Time*, Chap. 3, Cambridge Univ. Press, Cambridge 1973.

[2] J.A. Smoller, `gr-qc/0202057`.

[3] G. Magnano, L.M. Sokołowski, *Gen. Relativ. Gravitation* **30**, 1281 (1998).

[4] S. Weinberg, *Gravitation and Cosmology*, J. Wiley, New York 1972, Chap. 12; *The Quantum Theory of Fields*, vol. I, Chap. 7, Cambridge Univ. Press, Cambridge 1995.

[5] D.B. Fairlie, J. Govaerts, *J. Math. Phys.* **33**, 3543 (1992).

[6] D. Birmingham, M. Blau, M. Rakowski, G. Thompson, *Phys. Rep.* **209**, 129 (1991); A.S. Cattaneo, P. Cotta-Ramusino, J. Fröhlich, M. Martellini, *J. Math. Phys.* **36**, 6137 (1995); J. Baez, in *Geometry and Quantum Physics*, ed. by H. Gausterer and H. Grosse, Lecture Notes in Physics, vol. 543, Springer, Berlin 2000, `gr-qc/9905087`.

[7] L.D. Landau, E.M. Lifshitz, *The Classical Theory of Fields*, Chap. 3, Pergamon, New York 1980.

[8] G. Magnano, L.M. Sokołowski, *Phys. Rev.* **D50**, 5039 (1994).

[9] J.C. Alonso, F. Barbero, J. Julve, A. Tiemblo, *Class. Quantum Grav.* **11**, 865 (1994).

[10] G. Magnano, L.M. Sokołowski, *Ann. Phys. (N.Y.)* **306**, 1 (2003).

[11] J. Katz, J. Bičak, D. Lynden-Bell, *Phys. Rev.* **D55**, 5957 (1997).

[12] B. Julia, S. Silva, *Class. Quantum Grav.* **15**, 2173 (1998).

[13] A. Borowiec, M. Ferraris, M. Francaviglia, *J. Phys. A* **31**, 8823 (1998).

[14] J. Kijowski, W. Tulczyjew, *A Symplectic Framework for Field Theories*, Lecture Notes in Physics, vol. 107, Springer, Berlin 1979, Chap. 25.

[15] W. Kopczyński, *Ann. Phys. (N.Y.)* **203**, 308 (1990).

[16] F.W. Hehl, J.D. McCrea, E.W. Mielke, Y. Ne'eman, *Phys. Rep.* **258**, 1 (1995).

[17] A.N. Petrov, J. Katz, *Proc. R. Soc. Lond.* **A458**, 319 (2002).

[18] N.N. Bogoliubov, D.V. Shirkov, *Introduction to the Theory of Quantized Fields*, par. 2, Interscience, New York 1959.

[19] F.J. Belinfante, *Physica* **6**, 887 (1939); *Physica* **7**, 449 (1940).

[20] L. Rosenfeld, *Acad. Roy. Belg. Mem. Cl. Sci.* **18**, no. 6, 1 (1940).

[21] A. Trautman, in *Gravitation: An Introduction to Current Research*, ed. L. Witten, J. Wiley, New York 1962, p. 169.

[22] C. Aragone, S. Deser, *Nuovo Cim.* **57B**, 33 (1980).

[23] M. Ferraris, M. Francaviglia, *Class. Quantum Grav.* **9**, S79 (1992).

[24] D. Bak, D. Cangemi, R. Jackiw, *Phys. Rev.* **D49**, 5173 (1994).